The Galois group of the 27 lines on a rational cubic surface

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Cubic surfaces and lines

A *cubic surface* is the set of all points (x, y, z) satisfying a cubic polynomial equation.

In this talk we will be considering smooth cubic surfaces.

Example. The Fermat cubic is the set of all points (x, y, z) with

$$x^3 + y^3 + z^3 = -1.$$

Cubic surfaces and lines

The lines in the Fermat cubic $x^3 + y^3 + z^3 = -1$ are:

$$(-\zeta, t, -\omega t),$$

$$(-\zeta t, t, -\omega),$$

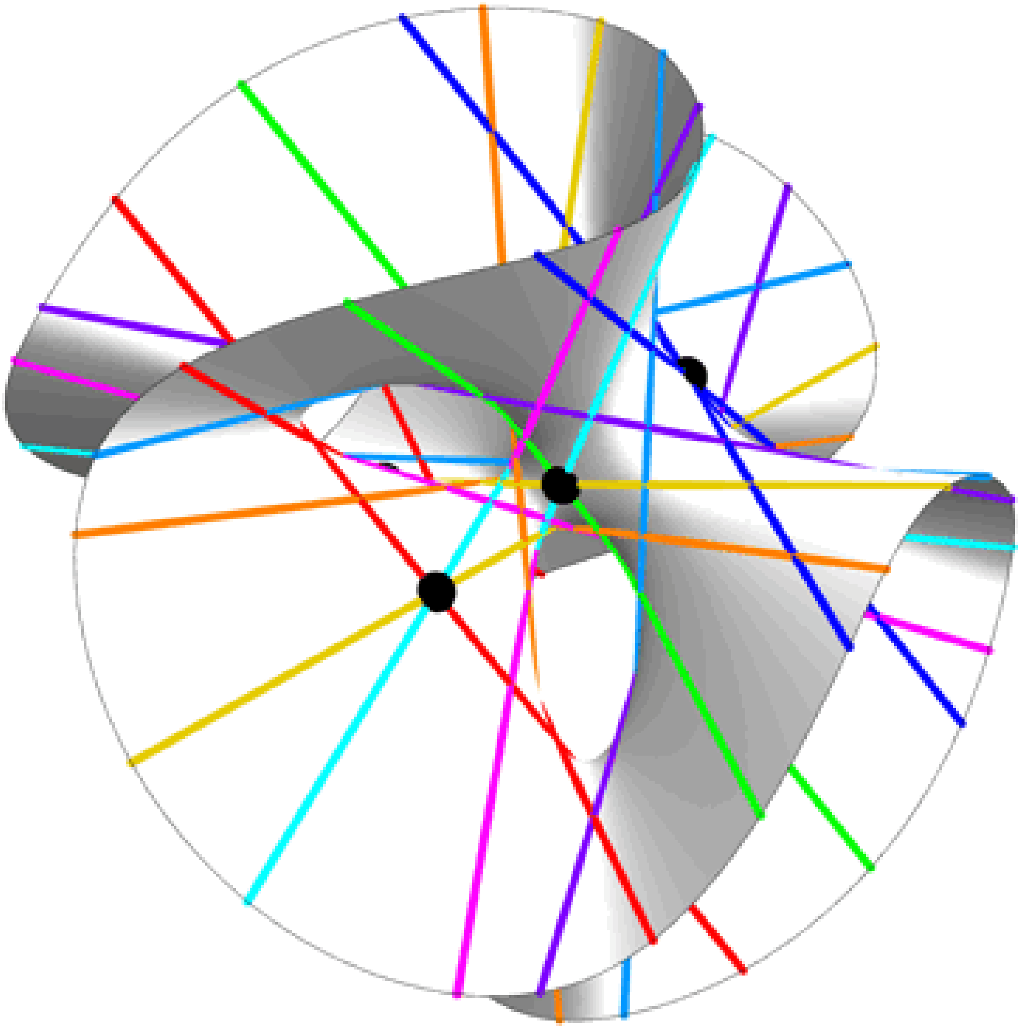
$$(-\zeta t, -\omega, t),$$

where t varies over $\mathbb{C}$, and ζ and ω are cube roots of 1.

Notice that there are 27 lines on this cubic surface.

Theorem. (*Cayley, 1849*) Every smooth cubic surface contains exactly 27 lines.

Cubic surfaces and lines



This is an example of a cubic surface.

Source: 27 Lines on a Cubic Surface by Greg Egan
http://www.gregegan.net/

Projective space

Define n-dimensional *projective space* $\mathbb{P}^n$ as the space of nonzero $(n+1)$-tuples of complex numbers $[x_0 : x_1 : x_2 : \ldots : x_n]$ up to scaling.

We may embed n-dimensional space $\mathbb{C}^n$ into projective space by setting one of the projective coordinates to 1.

Projective space can be considered as $\mathbb{C}^n$ with points at infinity.

A *line* in $\mathbb{P}^3$ is a linear embedding $\mathbb{P}^1 \hookrightarrow \mathbb{P}^3$. We can parametrize a general line in $\mathbb{P}^3$ as $[s : t : as + bt : cs + dt]$ for some fixed complex numbers $a, b, c,$ and d.

For the rest of the talk, we will be considering cubic surfaces in projective space.

The blow-up of $\mathbb{P}^2$ at six points

The *blow-up* of $\mathbb{P}^2$ at the origin is $\mathbb{P}^2 \setminus \{0\}$ with a line at 0, with each point on the line corresponding to a tangent direction at the point.

Define the blow-up of $\mathbb{P}^2$ at six points analogously.

We expect a general set of six points to be the zero locus of four linearly independent homogeneous polynomials in three variables.

If no three points are collinear and all six are not on a conic, this is true. Define the *generators* as these four cubic polynomials that vanish on these six points.

The blow-up of $\mathbb{P}^2$ at six points

Theorem. The blow-up of $\mathbb{P}^2$ at six points, no three collinear and all six not on a conic, is a cubic surface in $\mathbb{P}^3$. Every cubic surface can be obtained in this way.

Example. Let us blow-up $\mathbb{P}^2$ at the six points [1 : 1 : 0], [1 : –1 : 0], [1 : 0 : 1], [1 : 0 : –1], [0 : 1 : 1], and [1 : 1 : 1]. We may solve six linear equations to find the generators –

$$
\begin{gathered}
F_0 = x^3 - xy^2 - xz^2 + xyz, \\
F_1 = x^2y - y^3 + yz^2 - xyz, \\
F_2 = x^2z - z^3 + yz^2 - xyz, \\
F_3 = y^2z - yz^2.
\end{gathered}
$$

The blow-up of $\mathbb{P}^2$ at six points

The 27 lines on the corresponding cubic surface are –

- The preimages of the projections over the six points in $\mathbb{P}^2$, these are called *exceptional divisors*.
- The images of the fifteen lines through the six points under $[F_0 : F_1 : F_2 : F_3]$.
- The images of the six conics through the six points under $[F_0 : F_1 : F_2 : F_3]$.

Rational lines on a cubic surface

Let us work out one of the lines on the cubic surface in the previous example.

$$[F_0 : F_1 : F_2 : F_3] = [x^3 - xy^2 - xz^2 + xyz$$
$$: x^2y - y^3 - yz^2 - xyz : x^2z - z^3 + yz^2 - xyz : y^2z - yz^2].$$

We may plug in the line $z = 0$ (with parametrization $[s : t : 0]$) to get

$$[s^3 - st^2 : s^2t - t^3 : 0 : 0].$$

Thus $[s : t : 0 : 0]$ is a rational line on the cubic surface.

It turns out that all 27 lines on this cubic surface are rational.

Rational lines on a cubic surface

If a smooth cubic surface has rational generators, then it must have rational coefficients.

However, not all cubic surfaces with rational coefficients have rational lines or rational generators.

Question. Which field extension of $\mathbb{Q}$ is required to define the lines of a given rational cubic surface (that is, a cubic surface with rational coefficients)?

Field extensions

Consider the *automorphisms* of a field $K \supseteq \mathbb{Q}$. Note that these automorphisms must fix $\mathbb{Q}$, and that they form a group under composition.

This is the *Galois group* of $K/\mathbb{Q}$, denoted $\mathrm{Gal}(K/\mathbb{Q})$.

We can think of such fields K as vector spaces over $\mathbb{Q}$, and can define the *degree* $[K : \mathbb{Q}]$ as the dimension of this vector space. If K is finite, then the following holds:

Theorem. $\mathrm{Gal}(K/\mathbb{Q}) \leq S_{[K:\mathbb{Q}]}$, where S_n is the symmetric group.

The Galois group permutes the roots of any rational polynomial.

Galois groups of cubic surfaces

The group of permutations of the 27 lines, that preserves the relations between the lines, is the *Galois group* of X.

The *absolute Galois group* $G = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, for $\overline{\mathbb{Q}}$ the algebraic numbers, acts on the lines of any rational cubic surface X, and we may consider the subgroup H fixing the lines.

The quotient G/H is also the Galois group $\mathrm{Gal}(X)$ of X.

<u>Theorem</u>. (*Harris, 1979*) For a general cubic surface X, $\mathrm{Gal}(X) = W(E_6)$.

Question. What are the possible Galois groups of rational cubic surfaces?

Galois groups of cubic surfaces

The Galois group must be a subgroup of $W(E_6)$, it is the subgroup of permutations of the 27 lines that come from an element of the absolute Galois group.

If $K/\mathbb{Q}$ is the minimal extension required to define the lines of X, the Galois group of X is $\mathrm{Gal}(K/\mathbb{Q})$.

Example. The Galois group of a cubic surface obtained by blowing up $\mathbb{P}^2$ at six rational points is 1, as the minimal extension required to define the lines of X is $\mathbb{Q}$.

Galois groups of cubic surfaces

Example. If we blow up $\mathbb{P}^2$ at $[1:0:0]$ and $\left[1:\omega\sqrt[5]{2}:\omega^2\sqrt[5]{4}\right]$ for all 5th roots of unity ω, we have rational generators. We can verify that the corresponding cubic is given by

$$wy^2 - 2x^2y + wxz - z^3 = 0.$$

The minimal field to define the 27 lines is $\mathbb{Q}\left(\omega, \sqrt[5]{2}\right)$ (we can calculate the exceptional divisors to check this). Thus the Galois group is

$$\mathrm{Gal}(X) \cong \mathbb{Z}/_{5\mathbb{Z}} \rtimes \mathbb{Z}/_{4\mathbb{Z}},$$

where $\rtimes$ is the *semidirect product*.

Current work

Suppose that the generators are rational.

The six points being blown up must be invariant under the absolute Galois group, in other words, they are a union of *Galois orbits*. We then proved the following propositions.

Proposition. Let X be a smooth cubic surface with rational generators. Then the size of the Galois group of X divides 720.

This is because the Galois group of the minimal field to define an orbit of size k is a subgroup of S_k, and $|S_{i_1}||S_{i_2}|$ divides $|S_{i_1+i_2}|$.

Current work

We found the following interesting consequence –

A general cubic X of the form

$$w^3 + ax^3 + by^3 + cz^3 = 0$$

does not have rational generators.

We can explicitly compute the lines on X to see that

$$|\mathrm{Gal}(X)| = 54,$$

which is not a divisor of 720.

Current work

If the generators are irrational –

Proposition. Let X be a smooth cubic surface. If K is the minimal field required to define the generators, $[\mathrm{K} : \mathbb{Q}] \leq 72$.

Future work

We would like to answer the following questions:

- Which groups of size dividing 720 can be attained with rational generators?
- What happens in the case of irrational generators?

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